## The Natural Numbers

The number system developed as man needed some way to keep records of trading and other business transactions with his neighbors. The symbols 1, 2, 3,... which we use today are called the natural number system or the set of natural numbers. Note that the set of natural numbers, when considered as counting numbers, has no zero symbol. In early times there was no need for such a symbol.

As practical problems arose, addition and multiplication, which is repeated addition, developed naturally. The natural numbers were adequate for these operations.

The sum of two natural numbers is a natural number and the product of two natural numbers is a natural number. When an operation is performed on two numbers, of a set of numbers, and the result is a number of that set, the operation is said to have closure. In the set of natural numbers, consider the statement $3 + 4 = 7$. The sum of the natural numbers 3 and 4 is 7, a natural number. Now consider $3 \times 4 = 12$. The product of the natural numbers 3 and 4 is 12, a natural number. Thus, closure is preserved with respect to addition and multiplication in the set of natural numbers.

When the operations of subtraction and division were performed with natural numbers, the results were not always natural numbers. For example, $4 - 5 = -1$ and $4 \div 5 = \frac{4}{5}$. These results are not natural numbers. Thus closure is not preserved.

## The Integers

To preserve closure in subtraction the set of natural numbers was extended to include zero and negative numbers. The set of natural numbers with the addition of zero and negative numbers formed the set of integers. In the set of integers the operation of subtraction had closure.

## The Rational Numbers

Maintaining closure in division in the set of integers led to the invention of the rational number system. A rational number is one which can be expressed as the quotient of two integers, as $\frac{a}{b}$, with b not zero. The collection of all such pairs of numbers is called the rational number system. The rational number system or the set of rational numbers, includes the set of integers and the set of natural numbers.

## Irrational Numbers

The number system was again extended when man learned to extract roots (evolution). The roots of some positive rational numbers were positive rational numbers, while the roots of other positive natural numbers were not rational numbers. These roots were called irrational numbers. Examples of irrational numbers are $\sqrt{2}$, $\sqrt{5}$, $\sqrt{17}$.

## Real Numbers

The set of irrational numbers with the set of rational numbers form the system of real numbers or the set of real numbers.

## Properties of Equality of Real Numbers

**Reflexive property**

$a = a$

**Symmetric property**

If $a = b$, then $b = a$

**Transitive property**

If $a = b$, and $b = c$, then $a = c$

## Properties of Operations of Real Numbers

**Closure property**

If a and b are real numbers and $a + b = c$, c is also a real number. If $a \times b = ab$, ab is a real number.

**Commutative property**

$a + b = b + a$
$a \times b = b \times a$

**Associative property**

$a + b + c = (a + b) + c = a + (b + c)$
$a \times b \times c = a \cdot (b \times c) = (a \times b)\ c$

**Distributive property**

$a\ (b + c) = a \times b + a \times c = ab + ac$
$ab + ac = a\ (b + c)$

**Identity property**

$a + 0 = 0 + a = a$
$a \times 1 = 1 \times a = a$

**Inverse property**

$a \times \frac{1}{a} = \frac{1}{a} \times a = 1$, for $a = 0$

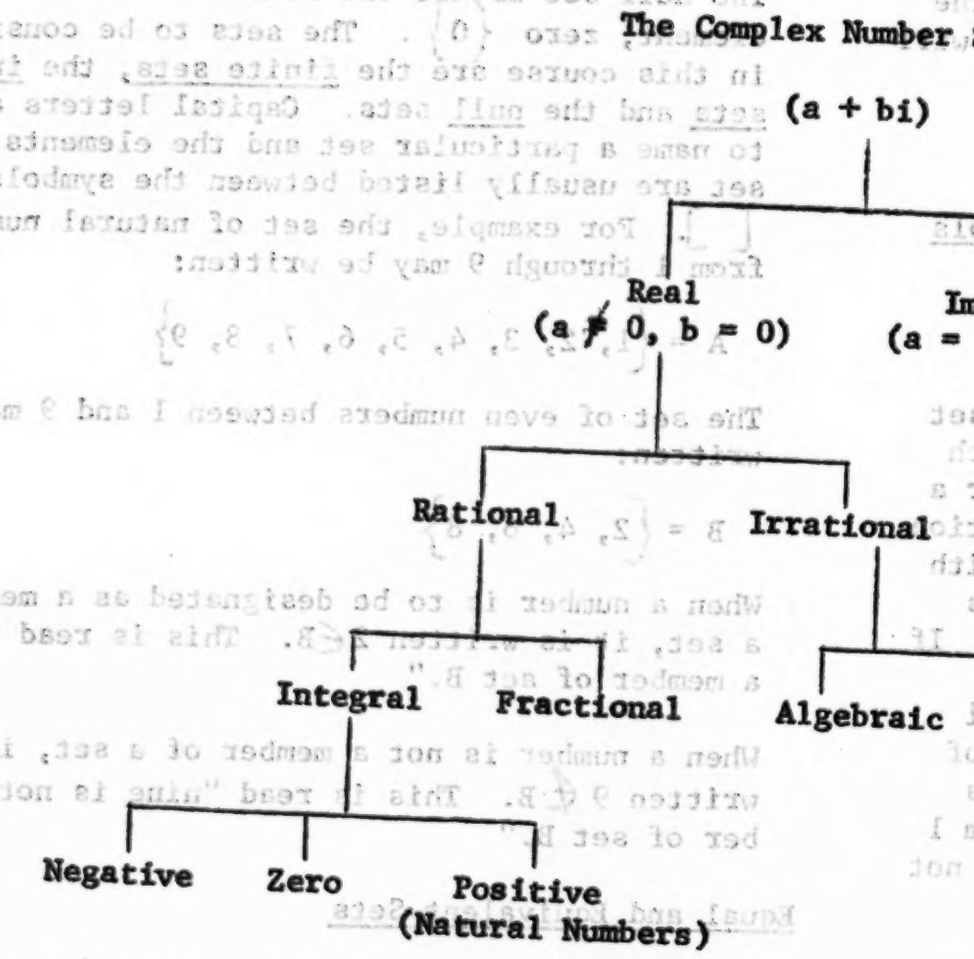
$a + (-a) = (-a) + a = 0$

## Complex Numbers

The set of complex numbers include both algebraic and non-algebraic numbers. An algebraic number is one which may be the root of an algebraic equation, in which the coefficients of its terms are rational numbers and the exponents are positive integers. Irrational roots of such equations are algebraic numbers and form a very large set. In fact there are more irrational roots than rational roots in the ordinary problems.

Irrational numbers that are not roots of algebraic equations are called transcendental numbers. Examples of transcendental numbers, in common usage, are $\pi$ and $e$.

The following diagram shows the relationship between the different kinds of numbers:


The Complex Number System
(a + bi)
Real
(a ≠ 0, b = 0)
Imaginary
(a = 0, b ≠ 0)
Rational
Irrational
Integral
Fractional
Algebraic
Transcendentals
Negative
Zero
Positive
(Natural Numbers)
Algebraic numbers
Non-algebraic
numbers

In the elementary algebra course, only the operations with the set of real numbers will be emphasized.

## SETS

### Basic Concept of Sets and Meaning of Symbols for Sets

#### Mathematical meaning of sets

The basic idea of set is not new. A set is simply a collection of objects, such as a set of dishes, a set of books, or a set of numbers. If the set or collection of dishes include only those dishes with a floral pattern, then all other types of dishes are not members of the set. If the collection of books includes only reference books in chemistry, then all other kinds of books are not members of the set. If the collection of numbers includes only the natural numbers from 1 through 9, then all other numbers are not members of the set.

Each object in the set is called a member or element of the set. A set may be very small or very large, depending on the number of its elements. If the number of elements of a set is countable, it is a finite set. If the number of elements of a set is not countable, it is an infinite set. If the set has no members, it is the empty set or null set. The empty set or null set is designated by the symbols $\emptyset$ or $\{\}$.

The null set is not the same as a set with one element, zero $\{0\}$. The sets to be considered in this course are the finite sets, the infinite sets and the null sets. Capital letters are used to name a particular set and the elements of the set are usually listed between the symbols $\{\}$ or $[\,]$. For example, the set of natural numbers from 1 through 9 may be written:

$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

The set of even numbers between 1 and 9 may be written:

$$B = \{2, 4, 6, 8\}$$

When a number is to be designated as a member of a set, it is written $2 \in B$. This is read "two is a member of set B."

When a number is not a member of a set, it is written $9 \notin B$. This is read "nine is not a member of set B."

### Equal and Equivalent Sets

The set of the first four letters of the alphabet may be written:

$$C = \{a, b, c, d\}$$

Note that set B has four members and set C also has four members. When two sets have the same number of members, they are said to be equivalent sets. In symbol form $C \leftrightarrow B$ and $B \leftrightarrow C$.

Let $D = \{8, 6, 2, 4\}$

Note that all the members of set B are also members of set D and both sets have the same number of elements. Sets B and D are called equal sets. Any two sets which have exactly the same members are equal sets. B = D.

If two sets are equivalent, each member of one set may be matched with a member of the equivalent set.

Since C$\leftrightarrow$B, the elements of C can be matched with the elements of B.

| | | |
|---|---|---|
| a$\leftrightarrow$2 | | a$\leftrightarrow$4 |
| b$\leftrightarrow$4 | or | b$\leftrightarrow$6 |
| c$\leftrightarrow$6 | | c$\leftrightarrow$8 |
| d$\leftrightarrow$8 | | d$\leftrightarrow$2 |

There are several ways in which the elements of C can be paired with the elements of B. Each time the elements of the two sets are paired, no elements in either set are left. When the elements in two sets can be matched or paired in this way, the sets are said to have a one-to-one correspondence.

## Subsets

If every member of a set, A, is also a member of a set, B, then A is a subset of B. This is written: A$\subset$B. If B has more members than A, then A is a proper subset of B.

If every member of A is also a member of B, A$\subset$B, and if every member of B is also a member of A, B$\subset$A, then A = B. If this is true of a given set M, then M is a subset of itself, M$\subset$M. In general, every set is a subset of itself. Also, the null set or empty set is a subset of every set.

## Universal Set

A set is understood to be a collection of objects which have some common characteristics. For example, set A may be the odd numbers from 1 through 11. A = {1, 3, 5, 7, 9, 11}. Set A is a subset of all the numbers {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11.} It is also a subset of the natural numbers. A larger set of a particular set is called the universal set or the universe and is designated by a capital U. In this example U = the natural numbers. All members of the infinite set {1, 2, 3, 4,...} are in this set.

## Operations with Sets

### Diagrams

Pictures of a set and its subsets, or its relation with other sets are called diagrams. A rectangle is usually used for the universal set and circles for the subsets.

Figure 1 represents all the members of the universal set.

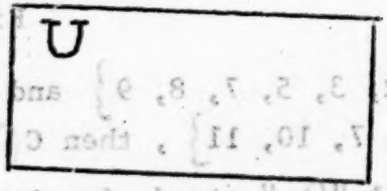



Figure 1

Figure 2 represents $A \subset U$. Set A is a proper subset of U.

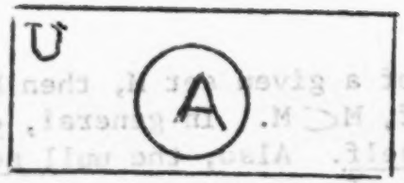



Figure 2

Figure 3 represents $A \subset U$ and $B \subset U$.

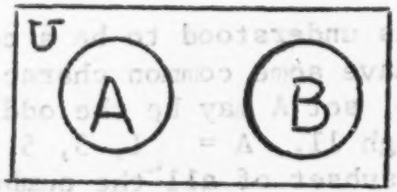



Figure 3

When two sets have no members in common, the sets do not intersect. Set A is <u>disjoint</u> from set B. Thus, the intersection of Set A and Set B is the null set.

## Intersection and Union of Sets

Figure 4 represents the intersection of C and D. All the elements in the shaded part of the diagram contain the elements common to sets C and D.

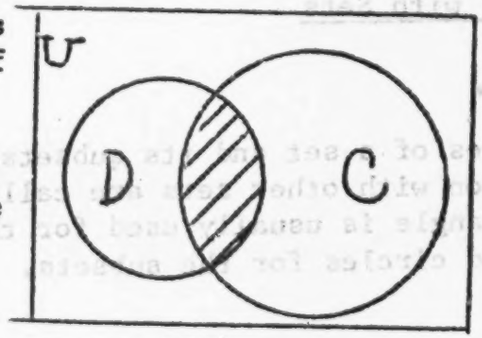



Figure 4

If $C = \{2, 3, 5, 7, 8, 9\}$ and $D = \{3, 5, 7, 10, 11\}$, then $C \cap B = \{3, 5, 7\}$.

The symbol "$\cap$" stands for <u>intersection</u>. It is also called <u>cap</u>.

In Figure 3, $A \cap B = \emptyset$ or $A \cap B =$

In Figure 5, the intersection of E and F is the set of one element.

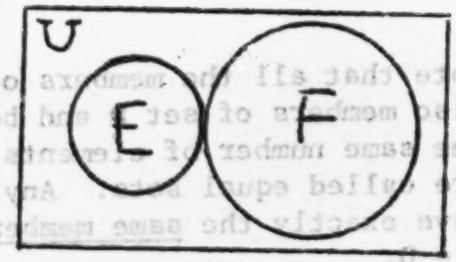



Figure 5

If $E = \{3, 4, 5\}$ and $F = \{1, 2, 3, 6, 7, 8\}$, then $E \cap F = \{3\}$.

In Figure 6, all the elements in U which are elements of at least one of the sets G or H form the set which is the <u>union</u> of G and H. The symbol $\cup$ is read <u>union</u>.

$G \cup H$ is the shaded portion of Figure 6.

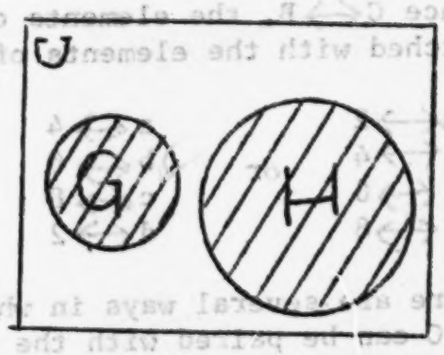



Figure 6

If $G = \{2, 4, 6\}$ and $H = \{3, 5, 7, 9\}$, then $G \cup H = \{2, 3, 4, 5, 6, 7, 9\}$. The sets are disjoint.

In Figure 7 if $K = \{1, 2, 3, 4, 5\}$ and $L = \{3, 4, 5, 8, 9, 10, 11\}$, then $K \cup L = \{1, 2, 3, 4, 5, 8, 9, 10, 11\}$

The sets intersect.

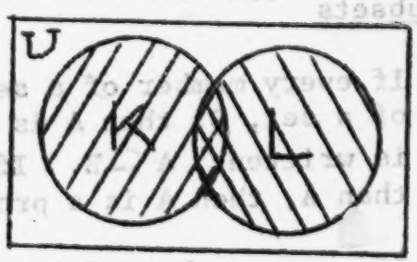



Figure 7

Note that the elements 3, 4, 5 are in both sets K and L, but they appear only once in the set formed by the union of sets K and L.

Figure 8 shows the intersection of sets M and N when N is a proper subset of M.

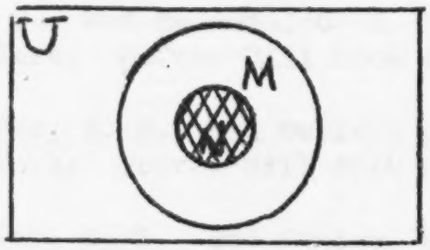



Figure 8

If $N \subset M$,
then $N \cap M = N$
and $N \cup M = M$

## Complementary Sets

Consider a set A in the universal set U of natural numbers. All elements in U, which are not elements in A, form the complementary set of A. It is written several ways, all having the same meaning.

Three of these ways are $\overline{A}$, $A'$, $\sim A$. All of these are read "the complement of set A." It follows that $U - A = A'$. The expression $U - A = A'$ means that when the elements of set A are removed from set U, the remaining elements form a new set, $A'$, the complement of set A. U - A is not a subtraction problem.

A and $A'$ are disjoint.

$A \cap A' = \emptyset$
$A \cup A = U$

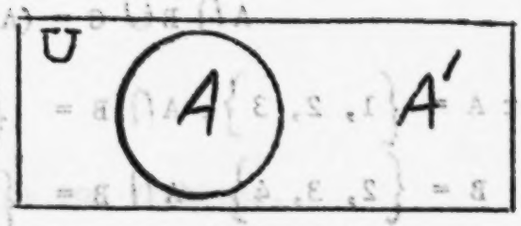



Figure 9

## Solving Problems with Sets

The fundamental operations of addition, subtraction, multiplication, and division are used in arithmetic to solve problems and obtain results. The sum or product with respect to the natural number systems is a natural number. Closure is preserved. The fundamental operations used with sets to solve problems and obtain results are union and intersection. The result obtained by the operation of intersection or union of two sets is another set with members in the universal set. Closure is preserved.

In arithmetic, to find the sum of 3 + 4 + 5, the sum of 3 and 4, (7), is found first and then the sum of 7 and 5, (12). Only two numbers can be added at a time. The same is true with multiplication. To find the product of 3 x 4 x 5, the product of 3 and 4, (12), is found first and then the product of 12 x 5, (60). In addition or multiplication, the operation is performed on only two numbers at a time. Addition and multiplication are binary operations.

The same is true when working with sets. If three sets, A, B, C, are involved in an operation or operations, the result with two sets is found first. Then the final answer is found by the operation on the first result and the third set.

$$A \cap B \cup C = (A \cap B) \cup C$$

Let $A = \{1, 2, 3\}$ $A \cap B = \{1, 2, 3\} \cap \{2, 3, 4\}$

$B = \{2, 3, 4\}$ $A \cap B = \{2, 3\}$

$C = \{1, 5, 6\}$ $(A \cap B) \cup C = \{2, 3\} \cup \{1, 5, 6\}$

$$(A \cap B) \cup C = \{1, 2, 3, 5, 6\}$$

Note that the elements 3, 4, 5 are in both sets K and L, but they appear only once in the set formed by the union of sets K and L.

Figure 8 shows the intersection of sets M and N when N is a proper subset of M.

If $N \subset M$,
then $N \cap M = N$
and $N \cup M = M$

Figure 8

Complementary Sets

Consider a set A in the universal set U of natural numbers. All elements in U, which are not elements in A, form the complementary set of A. It is written several ways, all having the same meaning.

$$A', \bar{A}, \complement A$$

Three of these ways are $A'$, $\bar{A}$, $\complement A$. All of these are read "the complement of set A." It follows that $U - A = A'$. The expression $U - A = A'$ means that when the elements of set A are removed from set U, the remaining elements form a new set, $A'$, the complement of set A. $U - A$ is not a subtraction problem.

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